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Error estimates for rotated Q_1^{rot} element approximation of the eigenvalue problem on anisotropic meshes ☆

Dongyang Shi [a], Yucheng Peng [a,b,*], Shaochun Chen [a]

[a] Department of Mathematics, Zhengzhou University, Zhengzhou, 450052, Henan, PR China

[b] Department of Mathematics, Xinyang Normal University, Xinyang 464000, Henan, PR China

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ABSTRACT

The main object of this work is to study the approximate behavior of the nonconforming rotated Q_1^{rot} element for the second-order elliptic eigenvalue problem on anisotropic meshes. A special technique is employed to construct a function possessing the anisotropic property in rotated Q_1^{rot} space, which leads to the optimal errors of energy norm and L^2 norm for the second-order elliptic boundary problem. The above results are then applied to the error analysis of eigenpairs and the associated optimal errors are derived. Numerical results are provided to show the validity of the theoretical analysis.

1. Introduction

It is well known that, for the nonconforming finite element, $|u-u_h|_h$, the error between the exact solution u and its approximation u_h, in energy norm, consists of the interpolation error and the consistency error, by the Strang lemma [1]. As usual, $\inf_{v\in V_h}|u-v|_h$ is controlled by $|u-Iu|_h$, and the optimal error estimation can then be derived using the Bramble–Hilbert lemma if $Ip=p$ for any $p\in P_k$, where I is the interpolation operator and P_k represents all the polynomials with degree no more than k, and V_h is the finite element space. Unfortunately, the above analysis relies on the regularity or quasi-uniformity assumption for meshes [1], i.e., there exists a constant $c>0$ such that for all elements R, $h_R/\rho_R\le c$, or $h/\widetilde{h}\le c$, where $h=\max_R h_R$, $\widetilde{h}=\min_R h_R$, h_R and ρ_R are the diameter and the supremum of the largest inscribed circle in R respectively. However, the domain considered may be narrow or irregular. For example, in modeling a gap between the rotor and stator in an electrical machine, or in modeling cartilage between a joint and the hip, if we seek the approximate solution with numerical methods by employing a regular partition on the domain, then the computing cost will be very high or it cannot be dealt with. It is better to employ the anisotropic subdivision which has fewer degrees of freedom than the traditional subdivision. Also, the solution of some elliptic boundary problems may generate sharp boundaries or interior layers, which means that the solution varies significantly in a certain direction. Examples include diffusion problems in domains with edges and singularly perturbed convention–diffusion–reaction problems. In such cases it is better to reflect this anisotropy in the discretization by using anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a large mesh size in the perpendicular direction. That is to say, the above regularity assumption or quasi-uniformity assumption is no longer valid and either h_R/ρ_R or $h/\widetilde{h}$ may be very large or even infinite. Consequently some basic theories and techniques of the classical finite element methods cannot be applied directly. For example, when the consistency error of a nonconforming element is estimated with traditional techniques, $\frac{meas(F)}{meas(R)}$ will appear and may be

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* Corresponding author at: Department of Mathematics, Zhengzhou University, Zhengzhou, 450052, Henan, PR China.
E-mail address: pyc111000@sina.com (Y. Peng).

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infinite if F is the longest edge of R. Therefore, novel techniques must be developed in order to obtain the error estimate in such cases. On the other hand, the Sobolev interpolation theory or Bramble–Hilbert lemma cannot be used directly on anisotropic meshes and a counterexample is given for the nonconforming rotated Q_1^{rot} element by Apel [2]. Thus, it becomes very difficult to check the anisotropy and the stability of interpolation operators on anisotropic meshes, and some basic theories used for checking the anisotropy have been given by Apel [3,4] and Chen et al. [5].

$\inf_{v\in V_h}|u-v|_h$ differs considerably from $|u-Iu|_h$; is there any element $v\in V_h$ such that $|u-v|_h$ has the same order of accuracy as $|u-Iu|_h$ and $|u-v|_h$ can be estimated on anisotropic meshes if the interpolation operator I does not possess anisotropy? Such an approach seems not have been studied in previous literature. So it is interesting to answer such a question from both practical computation and theoretical analysis points of view.

The purpose of this work is to study the approximate behavior of the nonconforming rotated Q_1^{rot} element for the second-order elliptic eigenvalue problem on anisotropic meshes. In Section 2, a special technique is developed for constructing a function possessing the anisotropic property in rotated Q_1^{rot} space, which leads to the optimal errors of $|u-u_h|_h$ and $\|u-u_h\|_0$ for the second-order elliptic boundary problem. This means that the theories proposed by Apel and Chen et al. for checking the anisotropy of an element are all sufficient conditions. In Section 3, the above results are then applied to the error analysis of eigenpairs for the second-order elliptic eigenvalue problem. Thus we improve the results of [2–4,8–10]. Finally, some numerical results are provided to demonstrate the validity of our theoretical analysis.

2. A special interpolation operator and its anisotropy

For the sake of simplicity, let $\Omega\subset R^2$ be a rectangle domain with boundary $\partial\Omega$ parallel to the x-axis or y-axis in the plane, and T_h be a family of axis-parallel rectangular meshes of Ω which does not need to satisfy the regularity condition or quasi-uniformity assumption. For any given rectangle $R\in T_h$, denote the barycentre of R by (x_R,y_R), the vertices by d_i, the sides by $l_i=\overline{d_id_{(i+1)(\mathrm{mod}\ 4)}}$ $(i=1,2,3,4)$, and the lengths of edges parallel to the x-axis and y-axis by $2h_x$ and $2h_y$, respectively.

Let $\widehat{R}=[-1,1]^2$ be a reference element in the ξ–η plane with vertices $\widehat{d}_1=(-1,-1)$, $\widehat{d}_2=(1,-1)$, $\widehat{d}_3=(1,1)$ and $\widehat{d}_4=(-1,1)$, $\widehat{l}_i=\overline{\widehat{d}_i\widehat{d}}_{(i+1)(\mathrm{mod}\ 4)}$ be the four edges of $\widehat{R}$. The affine transformation $F_R:\widehat{R}\longrightarrow R$ is defined by

$$x=x_R+h_x\xi,\qquad y=y_R+h_y\eta. \tag{1}$$

On $\widehat{R}$, the nonconforming rotated Q_1^{rot} element and the bilinear element $(\widehat{R},\widehat{Q}_1^{rot},\widehat{\Sigma}_1)$ [11] and $(\widehat{R},\widehat{Q}_2,\widehat{\Sigma}_2)$ [1] are defined as follows, respectively:

$$\widehat{Q}_1^{rot}=\{1,\xi,\eta,\xi^2-\eta^2\},\qquad \widehat{\Sigma}_1=\{\widehat{\phi}_i^1,i=1,\sim,4\}, \tag{2}$$

$$\widehat{Q}_2=\{1,\xi,\eta,\xi\eta\},\qquad \widehat{\Sigma}_2=\{\widehat{\phi}_i^0,i=1,\sim,4\}, \tag{3}$$

where $\widehat{\phi}_i^1=\frac{1}{|l_i|}\int_{\widehat{l}_i}\widehat{\phi}\,\mathrm{d}\widehat{s}$, $\widehat{\phi}_i^0=\widehat{\phi}(\widehat{d}_i)$.

One can check that for any $\widehat{\phi}\in H^2(\widehat{R})$, interpolation operators $\widehat{I}:H^2(\widehat{R})\longrightarrow\widehat{Q}_1^{rot}$ and $\widehat{\pi}:H^2(\widehat{R})\longrightarrow\widehat{Q}_2$ can be expressed as

$$\widehat{I}\widehat{\phi}=\frac14\sum_{i=1}^4\widehat{\phi}_i^1+\frac12(\widehat{\phi}_2^1-\widehat{\phi}_4^1)\xi+\frac12(\widehat{\phi}_3^1-\widehat{\phi}_1^1)\eta+\frac38(\widehat{\phi}_2^1+\widehat{\phi}_4^1-\widehat{\phi}_1^1-\widehat{\phi}_3^1)(\xi^2-\eta^2) \tag{4}$$

and

$$\widehat{\pi}\widehat{\phi}=\frac14\sum_{i=1}^4\widehat{\phi}_i^0+\frac14\left(\sum_{i=1}^4\xi_i\widehat{\phi}_i^0\right)\xi+\frac14\left(\sum_{i=1}^4\eta_i\widehat{\phi}_i^0\right)\eta+\frac14\left(\sum_{i=1}^4\xi_i\eta_i\widehat{\phi}_i^0\right)\xi\eta, \tag{5}$$

where $(\xi_1,\xi_2,\xi_3,\xi_4)=(-1,1,1,-1)$, $(\eta_1,\eta_2,\eta_3,\eta_4)=(-1,-1,1,1)$.

A lowest order quadrilateral nonconforming element $(\widehat{R},\widehat{P},\widehat{\Sigma})$ on $\widehat{R}$ proposed by Park and Sheen in [6] reads as

$$\widehat{P}=\{1,\xi,\eta\},\qquad \widehat{\Sigma}=\{\widehat{\phi}_i^1,i=1,\sim,4\}. \tag{6}$$

On the quadrilateral R, the following notation is set:

$$Q_1^{rot}=\{\phi=\widehat{\phi}\circ F_R^{-1},\widehat{\phi}\in\widehat{Q}_1^{rot}\},\qquad Q_2=\{\phi=\widehat{\phi}\circ F_R^{-1},\widehat{\phi}\in\widehat{Q}_2\},\qquad P=\{\phi=\widehat{\phi}\circ F_R^{-1},\widehat{\phi}\in\widehat{P}\}$$

and the corresponding interpolations of $v\in H^2(R)$ about Q_1^{rot} and Q_2 are

$$Iv=(\widehat{I}\widehat{v})\circ F_R^{-1},\qquad \pi v=(\widehat{\pi}\widehat{v})\circ F_R^{-1}.$$

The associated finite element spaces are

$$V_h^1=\left\{\phi;\phi|_R\in Q_1^{rot},\forall R\in T_h;\int_F[\phi]\mathrm{d}s=0,F\subset\partial R\right\},\qquad V_h^2=\{\phi;\phi|_R\in Q_2;\phi|_{\partial\Omega}=0\},$$

where $[\phi]$ denotes the jump of ϕ across the edge F of R if F is an internal edge, and $[\phi]=\phi$ if $F\subset\partial\Omega$.

Now one finite element space is defined as

$$V_h^3=\{\phi,\phi\in V_h,\phi|_R \text{ satisfies } \phi_1^1+\phi_3^1=\phi_2^1+\phi_4^1,\phi_i^1=\widehat{\phi}_i^1\circ F_R^{-1}\ (i=1,\ldots,4)\}.$$

It can be checked that the associated interpolation operator $\widehat{\widetilde{I}}$: $\widehat{\phi}\in H^1(\widehat{R})\longrightarrow V_h^3$ satisfies

$$\widehat{\widetilde{I}}\,\widehat{\phi}=\frac{1}{4}\sum_{i=1}^{4}\widehat{\phi}_i^1+\frac{1}{2}(\widehat{\phi}_2^1-\widehat{\phi}_4^1)\xi+\frac{1}{2}(\widehat{\phi}_3^1-\widehat{\phi}_1^1)\eta. \tag{7}$$

Lemma 2.1. *There holds*

$$\widehat{\widetilde{I}}=\widehat{\pi}\circ\widehat{I}, \tag{8}$$

i.e., for any $\widehat{\phi}\in H^2(\widehat{R})$

$$\widehat{\widetilde{I}}\,\widehat{\phi}=\widehat{\pi}\circ\widehat{I}\widehat{\phi}=\widehat{I}(\widehat{\pi}(\widehat{\phi})). \tag{9}$$

Proof. By the definition of interpolation operators $\widehat{\pi}$ and $\widehat{I}$, (8) or (9) immediately follows from (4), (5) and (7). The proof is completed. □

Remark. It is obvious that for any $\widehat{\phi}\in H^2(\widehat{R})$, $\widehat{\pi}(\widehat{I}(\widehat{\phi}))\in\widehat{Q}_1^{rot}$.

Lemma 2.2. *For any* $\widehat{\phi}\in H^2(\widehat{R})$ *and* $\alpha=(\alpha_1,\alpha_2)$ *with* $|\alpha|=1$*, we have the following local estimation:*

$$|D^\alpha(\widehat{\phi}-\widehat{\widetilde{I}}\,\widehat{\phi})|_{0,\widehat{R}}\le c|D^\alpha\widehat{\phi}|_{1,\widehat{R}}. \tag{10}$$

Proof. In fact, for $\alpha=(1,0)$, we have

$$D^\alpha(\widehat{\widetilde{I}}\,\widehat{\phi})=\frac{1}{2}(\widehat{\phi}_2^1-\widehat{\phi}_4^1)=\frac{1}{meas(\widehat{R})}\int_{\widehat{R}}\frac{\partial\widehat{\phi}}{\partial\xi}\mathrm{d}\xi\mathrm{d}\eta.$$

(10) follows from the Bramble–Hilbert lemma. Similarly (10) is valid for $\alpha=(0,1)$. The proof is completed. □

One can verify that for any $V\in H^2(\Omega)$ there holds

$$\widetilde{I}v\in V_h^1\bigcap V_h^3. \tag{11}$$

Thus, $\widetilde{I}v$ instead of Iv can be used to estimate the interpolation error of the rotated Q_1^{rot} element.

3. Analysis of the approximation for the eigenvalue problem

From now on, the approximation of eigenpairs is considered. For the sake of simplicity, the following second-order model eigenvalue problem is considered:

$$\begin{cases}-\Delta u=\lambda u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases} \tag{12}$$

The associated weak form of (12) is finding $0\ne u\in H_0^1(\Omega)$, $\|u\|_0=1$ and $\lambda\in R$ such that

$$a(u,v)=\lambda(u,v)\quad\forall v\in H_0^1(\Omega), \tag{13}$$

where $a(w,v)=\int_\Omega\nabla w\nabla v\mathrm{d}x\mathrm{d}y$, $(w,v)=\int_\Omega wv\mathrm{d}x\mathrm{d}y$.

The rotated Q_1^{rot} element approximation of (13) is finding $0\ne u_h\in V_h^3$, $\|u_h\|_0=1$ and $\lambda_h\in R$ such that

$$a_h(u_h,v)=\lambda_h(u_h,v)\quad\forall v\in V_h^3, \tag{14}$$

where $a_h(w,v)=\sum_R\int_R\nabla w\nabla v\mathrm{d}x\mathrm{d}y$.

In order to obtain the error estimate of (14) to (13), we first consider the following problem: find $\Phi\in H_0^1(\Omega)$ such that

$$a(\Phi,\phi)=(f,\phi)\quad\forall\phi\in H_0^1(\Omega). \tag{15}$$

The rotated Q_1^{rot} element approximation of (15) is finding $\Phi_h\in V_h^3$ such that

$$a_h(\Phi_h,\phi)=(f,\phi)\quad\forall\phi\in V_h^3. \tag{16}$$

Lemma 3.1 (*Strang's Lemma [1]*). *Suppose Φ and Φ_h to be the solutions of (15) and (16) respectively. Then*

$$|\Phi - \Phi_h|_h \le c\left(\inf_{\phi\in V_h^3} |\Phi - \phi|_h + \sup_{\phi\in V_h^3} \frac{|a_h(\Phi,\phi) - f(\phi)|}{|\phi|_h}\right), \tag{17}$$

where $|\varphi|_h = a_h(\varphi,\varphi)^{\frac{1}{2}}$. Here and later, c denotes a positive constant, a real number independent of h_R, h_R/ρ_R and $h/\widetilde{h}$.

Lemma 3.2. *On anisotropic meshes, suppose $\Phi \in H^2(\Omega)\bigcap H_0^1(\Omega)$ is the solution of (15); then for any $\phi \in V_{1h}$, there holds*

$$\left|\sum_R \int_{\partial R} \frac{\partial \Phi}{\partial \mathbf{n}} \phi \mathrm{d}s\right| \le ch|\Phi|_2|\phi|_h, \tag{18}$$

where $\mathbf{n}$ is the unit outer vector normal to ∂R.

Proof. One can check that

$$\begin{aligned}\sum_R \int_{\partial R} \frac{\partial \Phi}{\partial \mathbf{n}}\phi \mathrm{d}s &= \sum_R \left\{\left(\int_{F_3} \frac{\partial \Phi}{\partial y}\phi \mathrm{d}x - \int_{F_1} \frac{\partial \Phi}{\partial y}\phi \mathrm{d}x\right) + \left(\int_{F_2} \frac{\partial \Phi}{\partial x}\phi \mathrm{d}y - \int_{F_4} \frac{\partial \Phi}{\partial x}\phi \mathrm{d}y\right)\right\}\\ &= \sum_R \left\{\left[\int_{F_3}\left(\frac{\partial \Phi}{\partial y} - P_0\left(\frac{\partial \Phi}{\partial y}\right)\right)(\phi - P_{03}(\phi))\mathrm{d}x - \int_{F_1}\left(\frac{\partial \Phi}{\partial y} - P_0\left(\frac{\partial \Phi}{\partial y}\right)\right)(\phi - P_{01}(\phi))\mathrm{d}x\right]\right.\\ &\quad \left. + \left[\int_{F_2}\left(\frac{\partial \Phi}{\partial x} - P_0\left(\frac{\partial \Phi}{\partial x}\right)\right)(\phi - P_{02}(\phi))\mathrm{d}y - \int_{F_4}\left(\frac{\partial \Phi}{\partial x} - P_0\left(\frac{\partial \Phi}{\partial x}\right)\right)(\phi - P_{04}(\phi))v\mathrm{d}y\right]\right\},\end{aligned}$$

where $P_0(\varphi) = \frac{1}{meas(R)}\int_R \varphi \mathrm{d}x\mathrm{d}y$, $P_{0i}(\varphi) = \frac{1}{meas(F_i)}\int_{F_i}\varphi \mathrm{d}s$ $(i = 1, \sim, 4)$. Let [10]

$$Lv = \frac{x - (x_K - h_x)}{2h_x}P_{02}v - \frac{x - (x_K + h_x)}{2h_x}P_{04}v, \qquad Nv = \frac{y - (y_K - h_y)}{2h_y}P_{02}v - \frac{y - (y_K + h_y)}{2h_y}P_{04}v.$$

Therefore

$$\begin{aligned}&\sum_R \left[\int_{F_3}\left(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y}\right)(v - P_{03}v)\mathrm{d}x - \int_{F_1}\left(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y}\right)(v - P_{01}v)\mathrm{d}x\right]\\ &\quad = \sum_R \int_{\partial R}\left[-\left(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y}\right)(v - Nv)\right]\mathrm{d}x = \sum_R \int_R \frac{\partial}{\partial y}\left[\left(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y}\right)(v - Nv)\right]\mathrm{d}x\mathrm{d}y\\ &\quad = \sum_R \int_R \left[\frac{\partial^2 u}{\partial^2 y}(v - Nv) + \left(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial y}\right]\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Similarly

$$\begin{aligned}&\sum_R \left[\int_{F_2}\left(\frac{\partial u}{\partial x} - P_0\frac{\partial u}{\partial x}\right)(v - P_{02}v)\mathrm{d}y - \int_{F_4}\left(\frac{\partial u}{\partial x} - P_0\frac{\partial u}{\partial x}\right)(v - P_{04}v)\mathrm{d}y\right]\\ &\quad = \sum_R \int_{\partial R}\left[\left(\frac{\partial u}{\partial x} - P_0\frac{\partial u}{\partial x}\right)(v - Lv)\right]\mathrm{d}y = \sum_R \int_R \frac{\partial}{\partial x}\left[\left(\frac{\partial u}{\partial x} - P_0\frac{\partial u}{\partial x}\right)(v - Lv)\right]\mathrm{d}x\mathrm{d}y\\ &\quad = \sum_R \int_R \left[\frac{\partial^2 u}{\partial^2 x}(v - Lv) + \left(\frac{\partial u}{\partial y} - P_0\frac{\partial u}{\partial y}\right)\frac{\partial v}{\partial x}\right]\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Then employing local interpolation theory yields the desired result (18). □

Theorem 3.3. *Under the hypotheses of Lemma 3.2, let Φ_h be the solution of (16); then*

$$|\Phi - \Phi_h|_h \le ch|\Phi|_2, \qquad \|\Phi - \Phi_h\|_0 \le ch^2|\Phi|_2. \tag{19}$$

Proof. Noticing that

$$\inf_{\phi\in V_h^3} |\Phi - \phi|_h \le |\Phi - \widetilde{I}\Phi|_h \le \left(\sum_R |\Phi - \widetilde{I}\Phi|_{1,R}^2\right)^{\frac{1}{2}}$$

and keeping in mind Lemma 2.2, we have

$$|\Phi-\widetilde{I}\Phi|_{1,R}^2=\int_R\left[\left(\frac{\partial(\Phi-\widetilde{I}\Phi)}{\partial x}\right)^2+\left(\frac{\partial(\Phi-\widetilde{I}\Phi)}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y\le ch^2|\Phi|_2^2. \tag{20}$$

This, together with (18) and Lemma 3.1, yields the error estimate of the first term of (19).

On the other hand, for any $\psi\in L^2(\Omega)$, let Ψ be the unique solution of the following Dirichlet problem:

$$\begin{cases}-\Delta\Psi=\psi, & \text{in } \Omega,\\ \Psi=0, & \text{on } \partial\Omega.\end{cases} \tag{21}$$

Then

$$\|\Psi\|_2\le c\|\psi\|_0. \tag{22}$$

Consequently, we have

$$\begin{aligned}(\Phi_h-\widetilde{I}\Phi,\psi)&=(\Phi_h-\widetilde{I}\Phi,-\Delta\Psi)\\
&=a_h(\Phi_h-\widetilde{I}\Phi,\Psi)-\sum_R\int_{\partial R}\frac{\partial\Psi}{\partial n}(\Phi_h-\widetilde{I}\Phi)\mathrm{d}s\\
&=a_h(\Phi_h-\Phi,\Psi)+a_h(\Phi-\widetilde{I}\Phi,\Psi)-\sum_R\int_{\partial R}\frac{\partial\Psi}{\partial n}(\Phi_h-\widetilde{I}\Phi)\mathrm{d}s\\
&=a_h(\Phi_h-\Phi,\Psi-\widetilde{I}\Psi)+a_h(\Phi-\widetilde{I}\Phi,\Psi-\widetilde{I}\Psi)\\
&\quad+a_h(\Phi_h-\Phi,\widetilde{I}\Psi)+a_h(\Phi-\widetilde{I}\Phi,\widetilde{I}\Psi)-\sum_R\int_{\partial R}\frac{\partial\Psi}{\partial n}(\Phi_h-\widetilde{I}\Phi)\mathrm{d}s\\
&=a_h(\Phi_h-\Phi,\Psi-\widetilde{I}\Psi)+a_h(\Phi-\widetilde{I}\Phi,\Psi-\widetilde{I}\Psi)\\
&\quad+\sum_R\int_{\partial R}\frac{\partial\Phi}{\partial n}(\widetilde{I}\Psi-\Psi)\mathrm{d}s-\sum_R\int_{\partial R}\frac{\partial\Psi}{\partial n}(\Phi_h-\widetilde{I}\Phi)\mathrm{d}s.\end{aligned} \tag{23}$$

The following inequality follows from (18), the first estimate of (19) and (23):

$$|(\Phi_h-\widetilde{I}\Phi,\psi)|\le ch^2|\Phi|_2|\Psi|_2. \tag{24}$$

The second estimate of (19) is derived from (21), (22) and (23). The proof is completed. □

Lemma 3.4 ([7]). *Let (λ,u) and (λ_h,u_h) be eigenpairs of (13) and (14), respectively, such that $\|u\|_0=\|u_h\|_0=1$, $\lambda_h\longrightarrow\lambda$. Then*

$$\frac{\lambda_h-\lambda}{\lambda}=\frac{1}{(u,u_h)}(\bar{u}-u_h,u),\qquad \|u-u_h\|_0\le\frac{1}{\lambda_h}\frac{\|\bar{u}-u_h\|_0}{d(\frac{1}{\lambda})}\left(1+\frac{1}{\lambda_h}\frac{\|\bar{u}-u_h\|_0}{d(\frac{1}{\lambda})}\right), \tag{25}$$

where $d(t)=\inf_{t_j\neq t}|t_j-t|$; $\bar{u}\in H_0^1(\Omega)$ such that $a(\bar{u},v)=\lambda_h(u_h,v)$, $\forall v\in H_0^1(\Omega)$.

Now we state the main result of this work.

Theorem 3.5. *Let (λ,u), (λ_h,u_h) be the eigenpairs of (13) and (14) respectively, and $u\in H^2(\Omega)$. Then on anisotropic meshes, the following error estimates hold:*

$$|\lambda-\lambda_h|\le ch^2,\qquad \|u-u_h\|_0\le ch^2,\qquad |u-u_h|_1\le ch. \tag{26}$$

Proof. Let $f=\lambda_h u_h$ and $\bar{u}\in H^2(\Omega)\bigcap H_0^1(\Omega)$; then $\|\bar{u}-u_h\|_0\le c(\lambda)h^2$. Thus the second estimate of (26) follows from (25) and the above estimate. Noticing that

$$(u,u_h)=(u,u)+(u,u-u_h)\ge 1-\|u\|_0\|u-u_h\|_0=1-\|u-u_h\|_0,$$

then there exists a positive number h_0 such that $(u,u_h)\ge\frac{1}{2}$ for any $0<h\le h_0$ and

$$|\lambda_h-\lambda|\le 2\|u-u_h\|_0\|u\|_0\le c(\lambda)h^2.$$

Also suppose $\widetilde{u}_h\in V_h^3$ satisfies

$$a_h(\widetilde{u}_h,v)=\lambda(u,v),\quad \forall v\in V_h^3.$$

Replacing f by λu in Theorem 3.3, we have

$$\|u-\widetilde{u}_h\|_0 + h|u-\widetilde{u}_h|_h \le ch^2|u|_2.$$

On the other hand,

$$\begin{aligned}|\widetilde{u}_h-u_h|_h^2 = a_h(\widetilde{u}_h-u_h,\widetilde{u}_h-u_h) &\le |(\lambda u,\widetilde{u}_h-u_h)-(\lambda_h u_h,\widetilde{u}_h-u_h)|\\ &= |(\lambda u-\lambda_h u_h,\widetilde{u}_h-u_h)|\\ &\le \|\lambda u-\lambda_h u_h\|_0\|\widetilde{u}_h-u_h\|_0.\end{aligned}$$

Furthermore

$$|\widetilde{u}_h-u_h|_h \le \|\lambda u-\lambda_h u_h\|_0 + \|\widetilde{u}_h-u_h\|_0 \le |\lambda|\|u-u_h\|_0 + |\lambda-\lambda_h|\|u_h\|_0 + \|\widetilde{u}_h-u\|_0 + \|u-u_h\|_0$$

and

$$|u-u_h|_h \le |u-\widetilde{u}_h|_h + |\widetilde{u}_h-u_h|_h \le ch.$$

The proof is completed. □

By the above analysis of the error between λ_h and λ, now we focus on higher convergence analysis of eigenvalues through Richardson extrapolation.

Lemma 3.6. *Under the condition of Theorem 3.5, there holds*

$$\lambda_h-\lambda = a_h(u-\widetilde{u}_h,u_h)+\lambda(u-\widetilde{I}u,u_h)-a_h(u-\widetilde{I}u,u_h)+O(h^4). \tag{27}$$

Proof. Because

$$(u,u_h)=(u,u)-(u,u_h-u)\ge 1-\|u-u_h\|_0 \ge 1-ch^2.$$

Let $\bar{u}_h = u_h/(u,u_h)$; then

$$\|u_h-\bar{u}_h\|_0 = |(u,u-u_h)u_h/(u,u_h)| \le ch^2.$$

This shows that u_h hardly differs from $\bar{u}_h$ in L^2-norm if h is small enough. On the other hand,

$$\begin{aligned}&\lambda_h=\lambda_h(u,\bar{u}_h)=\lambda_h(u-\widetilde{u}_h,\bar{u}_h)+\lambda_h(\widetilde{u}_h,\bar{u}_h),\\ &\lambda_h(\widetilde{u}_h,\bar{u}_h)=a_h(\widetilde{u}_h,\bar{u}_h)=\lambda(u,\bar{u}_h)=\lambda,\end{aligned}$$

so there holds

$$\begin{aligned}\lambda_h &= \lambda+\lambda_h(u-\widetilde{u}_h,\bar{u}_h)=\lambda+\lambda_h(u-\widetilde{u}_h,u_h)+\lambda_h(u-\widetilde{u}_h,\bar{u}_h-u_h)\\ &=\lambda+\lambda_h(u-\widetilde{I}u,u_h)+\lambda_h(\widetilde{I}u-\widetilde{u}_h,u_h)+\lambda_h(u-\widetilde{u}_h,\bar{u}_h-u_h)\\ &=\lambda+\lambda_h(u-\widetilde{I}u,u_h)+a_h(\widetilde{I}u-\widetilde{u}_h,u_h)+\lambda_h(u-\widetilde{u}_h,\bar{u}_h-u_h)\\ &=\lambda+\lambda(u-\widetilde{I}u,u_h)+a_h(\widetilde{I}u-u,u_h)+a_h(u-\widetilde{u}_h,u_h)+(\lambda_h-\lambda)(u-\widetilde{I}u,u_h)+\lambda_h(u-\widetilde{u}_h,\bar{u}_h-u_h),\end{aligned}$$

(27) thanks to Theorem 3.5. The proof is completed. □

Lemma 3.7. *Let $u\in H_0^1(\Omega)\bigcap H^4$, then for any $v\in V_h^3$, there hold*

$$\begin{cases}(u-\widetilde{I}u,v) = -\dfrac{1}{6}\displaystyle\sum_{R\in T_h}\left[h_x^2\int_R u_{xx}v\,\mathrm{d}x\mathrm{d}y+h_y^2\int_R u_{yy}v\,\mathrm{d}x\mathrm{d}y\right]+ch^3|u|_3|v|_h,\\ a_h(u-\widetilde{u}_h,v)=\dfrac{1}{3}\displaystyle\sum_{R\in T_h}\left\{h_x^2\int_R u_{xyy}v_x\,\mathrm{d}x\mathrm{d}y+h_y^2\int_R u_{xxy}v_y\,\mathrm{d}x\mathrm{d}y\right\}+ch^3|u|_4|v|_h,\\ a_h(u-\widetilde{I}u,v)=-\dfrac{1}{3}\displaystyle\sum_{R\in T_h}\left[h_y^2\int_R u_{xyy}v_x\,\mathrm{d}x\mathrm{d}y+h_x^2\int_R u_{xxy}v_y\,\mathrm{d}x\mathrm{d}y\right]+ch^3|u|_4|v|_h.\end{cases} \tag{28}$$

Proof. One can check that

$$\int_R (u-\widetilde{I}u)v\,\mathrm{d}x\mathrm{d}y+\frac{1}{6}h_x^2\int_R u_{xx}v\,\mathrm{d}x\mathrm{d}y+\frac{1}{6}h_y^2\int_R u_{yy}v\,\mathrm{d}x\mathrm{d}y=0$$

holds for any $u \in P_2(R)$ and $v \in V_h^3$. By the Bramble–Hilbert–Xu lemma, the first argument of (28) is obtained. Like for Lemma 3.2 one can get

$$a_h(u-\widetilde{u}_h,v) = a_h(u,v) - a_h(\widetilde{u}_h,v) = \sum_R \left[\left(\int_{F_3} - \int_{F_1}\right)\frac{\partial u}{\partial y} v\mathrm{d}x + \left(\int_{F_2} - \int_{F_4}\right)\frac{\partial u}{\partial x} v\mathrm{d}y\right]$$
$$= \sum_R \left\{\int_{F_3} \frac{\partial u}{\partial y}(v-v_3)\mathrm{d}x - \int_{F_1} \frac{\partial u}{\partial y}(v-v_1)\mathrm{d}x + \int_{F_2} \frac{\partial u}{\partial x}(v-v_2)\mathrm{d}y - \int_{F_4} \frac{\partial u}{\partial x}(v-v_4)\mathrm{d}y\right\}$$

and

$$\left(\int_{F_1} - \int_{F_3}\right)\frac{\partial u}{\partial y}(v-v_i)\mathrm{d}x = \int_R \frac{\partial^2 u}{\partial^2 y} v_x (x-x_R)\mathrm{d}x\mathrm{d}y.$$

For any $\phi \in P_1$ and $v \in V_h^3$, there holds

$$\int_R \phi v_x (x-x_R)\mathrm{d}x\mathrm{d}y - \frac{1}{3}h_x^2 \int_R \phi_x v_x \mathrm{d}x\mathrm{d}y = 0.$$

The Bramble–Hilbert–Xu lemma is used again to yield

$$\int_R \phi v_x (x-x_R)\mathrm{d}x\mathrm{d}y = \frac{1}{3}h_x^2 \int_R \phi_x v_x \mathrm{d}x\mathrm{d}y + ch^3|\phi|_{2,R}|v|_{1,R}.$$

Similarly

$$\int_R \phi v_y (y-y_R)\mathrm{d}x\mathrm{d}y = \frac{1}{3}h_y^2 \int_R \phi_y v_y \mathrm{d}x\mathrm{d}y + ch^3|\phi|_{2,R}|v|_{1,R}.$$

Therefore

$$a_h(u-\underline{u}_h,v) = \frac{1}{3}\sum_{R\in T_h}\left\{h_x^2 \int_R u_{xxy} v_x \mathrm{d}x\mathrm{d}y + h_y^2 \int_R u_{xyy} v_y \mathrm{d}x\mathrm{d}y\right\} + ch^3|u|_4|v|_h. \tag{29}$$

The last argument of this lemma can be obtained by the same technique. The proof is completed. □

Theorem 3.8. *Suppose (λ, u), (λ_h, u_h) are the eigenpairs of (13) and (14) respectively, and $u \in H_0^1(\Omega)\bigcap H^4(\Omega)$. Then on anisotropic meshes, the following error estimate:*

$$\lambda_h - \lambda = -\frac{\lambda}{6}\sum_{R\in T_h}\left[h_x^2\int_R u_{xx}u\mathrm{d}x\mathrm{d}y + h_y^2\int_R u_{yy}v\mathrm{d}x\mathrm{d}y\right]$$
$$+\frac{1}{3}\sum_{R\in T_h}\left\{(h_x^2+h_y^2)\left(\int_R u_{xyy}u_x\mathrm{d}x\mathrm{d}y + \int_R u_{xxy}u_y\mathrm{d}x\mathrm{d}y\right)\right\} + ch^3 \tag{30}$$

and the a posteriori error estimate obtained by the Richardson extrapolation technique

$$\frac{4\lambda_{\frac{h}{2}} - \lambda_h}{3} = \lambda + ch^3 \tag{31}$$

hold.

Proof. Replacing v with u_h in Lemma 3.7 and noticing that $\|u-u_h\|_0$, $|u-u_h|_h$ converge to zero as $h \longrightarrow 0$ yields (30). By the Richardson extrapolation technique, (31) follows from (30). The proof is completed. □

4. Numerical example

In order to verify our theoretical analysis and examine the performance of the element for eigenvalues under anisotropic meshes, the eigenvalue problem (12) is considered with $\Omega = [-1,1]\times[-,1]$, and the exact solution is $u = \sin(\alpha\pi x)\sin(\beta\pi y)$; hence the eigenvalue $\lambda_{\alpha,\beta} = (\alpha^2+\beta^2)\pi^2$.

The subdivisions are used on square meshes and anisotropic meshes with n segments along the x axis and m segments along the y axis, respectively. Tables 1 and 2 denote the finite element solution $\lambda_{\alpha,\beta}^h$ of the problem (14), their extrapolations $\chi_{\alpha,\beta}^h$ and the average convergence order δ. Let $\lambda_{\alpha,\beta}^h$ and $\lambda_{\alpha,\beta}^{h/2}$ denote the finite solution on $n\times m$ and $2n\times 2m$ meshes, respectively, $\mu_{\alpha,\beta}^h = (4\lambda_{\alpha,\beta}^{h/2} - \lambda_{\alpha,\beta}^h)/3$, $\chi_{\alpha,\beta}^i = \chi_{\alpha,\beta}^{h/2^{i-1}}$, $\varepsilon_i = |\lambda_{\alpha,\beta} - \chi_{\alpha,\beta}^i|$, $\delta_i = \log(\varepsilon_i/\varepsilon_{i+1})/\log 2\ (1\le i\le 3)$, $\delta = \sum_{i=1}^3 \delta_i/3$ (or $\delta = \sum_{i=1}^2 \delta_i/2$ for extrapolations).

Table 1
Approximation results for the $\lambda_{\alpha,\beta}$ and their extrapolations on square meshes respectively.

$n \times m$	8×8	16×16	32×32	64×64	δ
$\lambda_{1,1}^h$	1.973724e+01	1.973909e+01	1.973920e+01	1.973921e+01	4.008287
$\lambda_{2,1}^h$	5.049758e+01	4.963399e+01	4.941940e+01	4.936586e+01	2.000624
$\lambda_{1,2}^h$	5.049758e+01	4.963399e+01	4.941940e+01	4.936586e+01	2.000624
$\lambda_{2,2}^h$	7.882873e+01	7.894896e+01	7.895635e+01	7.895680e+01	4.026570
$\mu_{1,1}^h$	1.973971e+01	1.973924e+01	1.973921e+01		3.389865
$\mu_{2,1}^h$	4.934613e+01	4.934787e+01	4.934801e+01		3.885928
$\mu_{1,2}^h$	4.934613e+01	4.934787e+01	4.934801e+01		3.885928
$\mu_{2,2}^h$	7.898904e+01	7.895881e+01	7.895695e+01		4.065985

Table 2
Approximation results for the $\lambda_{\alpha,\beta}$ and their extrapolations on anisotropic meshes respectively.

$n \times m$	2×20	4×40	8×80	16×160	δ
$\lambda_{1,1}^h$	1.940272e+01	1.972234e+01	1.973821e+01	1.973915e+01	4.139889
$\lambda_{2,1}^h$	4.184036e+01	4.779016e+01	4.897683e+01	4.925633e+01	2.118494
$\lambda_{1,2}^h$	8.001244e+01	5.538844e+01	5.087847e+01	4.972850e+01	2.110866
$\lambda_{2,2}^h$	1.351211e+02	7.761082e+01	7.888934e+01	7.895285e+01	4.594058
$\mu_{1,1}^h$	1.982888e+01	1.974350e+01	1.973946e+01		4.230356
$\mu_{2,1}^h$	4.977343e+01	4.937239e+01	4.934950e+01		4.086154
$\mu_{1,2}^h$	4.718044e+01	4.937515e+01	4.934518e+01		4.786635
$\mu_{2,2}^h$	5.844073e+01	7.931551e+01	7.897402e+01		5.110705

The data in Tables 1 and 2 show that the finite solution for the eigenvalues converges to that of problem (12) on both square and anisotropic meshes. Notice that the convergence order is higher than that of theoretical analysis for most of eigenvalues. The explanation of this phenomenon may only be a special property of the nonconforming finite element which has not been discovered.

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